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DOMINATION IN (2,1)-FUZZY GRAPHS AND ITS APPLICATION

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Abstract. The notion of a fuzzy graph (FG) and its expanded variations have been devised to address several real-world issues including imprecision, such as decision-making, networking, shortest path, and so forth. The generalization of FG theory to circumstances where imprecision is characterized by differences in the values of membership and non-membership grades is the basis of this study. The purpose of this study is to present the (2,1)-fuzzy set graph concept. Furthermore, in a (2,1)-fuzzy environment, this study examines the idea of domination theory. More specifically, the theory related to (2,1)-fuzzy graphs is presented along with illustrative examples, introducing the framework. Additionally, the domination theory associated with (2,1)-fuzzy graphs is developed. Finally, a numeral example is presented to explain the computing of domination in (2,1)-fuzzy graph in the specific application.

Keywords: (2,1)-Fuzzy set, fuzzy graphs, (2,1)-fuzzy graphs, domination, dominating set.
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1 Introduction

The first research on fuzzy set theory, which was developed by Lotfi Zadeh, was published in 1965 (Zadeh, 1965). It handled uncertainty by assigning a degree of membership to each object in the set. However, in real life, we come into circumstances in which we are unable to judge if something is right or wrong. A useful technique for reasoning in these circumstances is fuzzy logic. It can thus stand for both the truth and falsity in any given circumstance. The absolute truth value in the Boolean system is represented by the truth value 1.0, whereas the absolute false value is represented by the value 0.0. There is an intermediate value in fuzzy logic, nevertheless, that is both partially true and partially false. Fuzzy logic and sets have widespread applications in real-world technology. In the domains of engineering, mathematics, computer science, medicine, business and economics, social sciences, and human behavior researches, for instance, it has been utilized to find efficient solutions under uncertainty.

A characteristic function in the unit range of 0 to 1, which specifies the degree of membership of an element in a specific set, is the fundamental foundation of a fuzzy set. The classic fuzzy set has a difficulty in that the degree of membership for an element only defines the degree of membership, whereas the degree of membership and the sum of the degree of non-membership (non-membership) equals one. As a result, fuzzy set theory is unable to account for subjective assessments of what constitutes discontent. This inspired Atanassov to expand on the notion of

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fuzzy sets and propose the notion of intuition fuzzy sets in order to get around this particular restriction (Atanassov, 1999). For intuitionistic fuzzy sets, Atanassov (1999) defined two new operators and looked into the essential characteristics and attributes of them. However, the sum of the membership and non-membership degrees in the currently available intuitive fuzzy sets cannot be greater than one, which is regarded as a limitation in certain situations.

In order to stay up with the latest discoveries, intuitive fuzzy sets need to be improved, as uncertainty is becoming a more difficult problem across a wide range of disciplines. Some authors have recently suggested using distinct definitions for degrees of membership and non-membership in order to handle input data. This method will help expand the scope of the data domains analyzed and find a few real-world issues. In this regard, Al-shami (2023) developed the fundamental set of operations for (2,1)-Fuzzy sets and introduced the concept.

Graph theory first aids in the most efficient representation, organization, and resolution of a task or problem that needs to be solved. For this reason, methodical approaches are looked for after a problem is converted into a graph structure in order to determine the quickest or most affordable manner to complete all of the problem's objectives. This naturally leads to a wide range of application areas for graph theory. The fact that graph theory is applicable to numerous other scientific fields is the primary factor for its development. For instance, by turning complicated problems in theoretical computer science into graph theory problems, most of the problems may be addressed more quickly and efficiently. In addition, graph theory is more valuable because mathematics and other scientific fields share a similar subject. Graph theory applications are used to address a wide range of intricate and comprehensive issues in contemporary life. These applications include disciplines like information transmission, marketing, sales and marketing, economics, and management science. Determining linkages architecturally and identifying problems are two further uses for graph theory.

Ore (1962) developed the notions of minimal and least dominating sets of vertices in a graph in 1962 and was the first to use the term "domination" for undirected graphs. Kaufmann (1973) gave the definition of fuzzy graph using fuzzy relation. In fuzzy graphs, domination and independent domination were introduced by Somasundaram and Somasundaram (1998). Strong arcs were used by Nagoorgani and Chandrasekaran (2006) to investigate the concepts of fuzzy domination and independent domination. In intuitionistic fuzzy graphs, edge domination was examined by Shubatah et al. (2020). Mahioub (2012) and Shubatah et al. (2012) studied product fuzzy graphs and domination.

A suitable framework for modeling certain real-world issues assessed by the varied significance of membership and non-membership notes is offered by the (2,1)-fuzzy notion. The idea of a (2,1)-fuzzy graph arises to remove boundary limitations and deal with numerous real-world issues. This paper aims to broaden the understanding of graph theory principles in a (2,1)-fuzzy environment. The main objective of this research is to define (2,1)-Fuzzy graphs, a new generalization of intuitionistic fuzzy graphs. In contrast to intuitionistic fuzzy graphs, this generalization extends the space of membership and non-membership degrees. Additionally, this study helps to acquire the relevant expanded notions of various (2,1)-fuzzy graphs and extends the concept of domination in the fuzzy graph to the (2,1)-fuzzy frameworks. In the meanwhile, methods for obtaining the particular dominating sets are introduced. Also, a domination application in a (2,1)-fuzzy graph modeling the circumstances of an election campaign is given.

2 Preliminaries

In this section, we recall some basic definitions of fuzzy set, fuzzy graph, intuitionistic graph and domination in graphs.

Definition 1. *Consider the universal set U. A fuzzy set F in U is represented by $F = \{(u, \mu_F(u)) : \mu_F(u) > 0, u \in U\}$, where the function $\mu_F : U \to [0,1]$ is the membership degree of element u in the fuzzy set F (Zadeh, 1965).*

Definition 2. *A fuzzy graph is a pair $G_F = (B, C)$, where $B = \{\mu_B\}$ and $C = \{\mu_C\}$ such that $\mu_B : W \to [0,1]$ and $\mu_C : W \times W \to [0,1]$. We have $\mu_C(v,z) \leq \mu_B(v) \wedge \mu_B(z)$, for all v,z $\in$ W (Rosenfeld, 1975).*

Definition 3. *A fuzzy graph $G^A = (B^A, C^A)$ is said to be a fuzzy subgraph of G, if $\mu^A(v) \leq \mu(v)$, $\mu^A(v,z) \leq \mu(v,z)$, for all $v, z \in W$ (Rosenfeld, 1975).*

Definition 4. *Let U be a non-empty set. An Intuitionistic Fuzzy set F in U is an object of the form $F = \{(u, \mu_F(u), \nu_F(u)) : u \in U\}$, where the function $\mu_F : U \to [0,1]$ and $\nu_F : U \to [0,1]$ determine the degree of membership and the degree of non-membership of the element $u \in U$, respectively and for every $u \in U$, $0 \leq \mu_F(u) + \nu_F(u) \leq 1$ (Atanassov, 1999).*

Definition 5. *Let W be a non-empty set. An Intuitionistic Fuzzy Graph is of the form $G^* = (W, Z)$, where $W = \{v_1, v_2, ..., v_n\}$ such that $\mu_B : W \to [0,1]$ and $\nu_B : W \to [0,1]$ denote the degree of membership and non-membership of the element $v_i \in W$ respectively and $0 \leq \mu_B(v_i) + \nu_B(v_i) \leq 1$, for every $v_i \in W (i = 1, 2, ..., n)$. $Z \subset W \times W$ where $\mu_C : W \times W \to [0,1]$ and $\nu_C : W \times W \to [0,1]$ such that*

$$\mu_C(v_i, v_j) \leq \mu_B(v_i) \wedge \mu_B(v_j)$$
$$\nu_C(v_i, v_j) \leq \nu_B(v_i) \vee \nu_B(v_j)$$

and $0 \leq \mu_C(v_i, v_j) + \nu_C(v_i, v_j) \leq 1$ for every $(v_i, v_j) \in Z$ (Parvathi and Karunambigai, 2006).

Definition 6. *The (2,1)-Fuzzy set D over the universal set U is defined as follows: $D = \{(u, \mu_D(u), \nu_D(u)) : u \in U\}$, where the functions μ_D and ν_D from U into [0,1] respectively represent the membership and non-membership degrees of every $u \in U$ to D under the constraint $0 \leq (\mu_D(u))^2 + \nu_D(u) \leq 1$ (Al-shami, 2023).*

Definition 7. *A dominating set S in $G^* = (W, Z)$ is defined as for every $v \in W - S$ dominates some $z \in S$, where $S \subseteq W$ (Somasundaram and Somasundaram, 1998).*

Definition 8. *An arc (v,z) is said to be a strong arc, if $\mu_C(v,z) \geq \mu_C^{\infty}(v,z)$ (Bhutani and Rosenfeld, 2003).*

Definition 9. *Let v,z $\in$ W. In the fuzzy graph $G_F = (B, C)$ introduced the validity of edge (v,z) $\in$ Z by*

$$I(v,z) = \frac{\mu_C(v,z)}{\mu_B(v) \wedge \mu_B(z)}.$$

Clearly, that $0 \leq I(v,z) \leq 1$. If $I(v,z) \geq \frac{1}{2}$, the edge (v,z) is called valid, if not, it is called invalid edge (Afsharmanesh and Borzooei, 2021).

3 (2,1)-Fuzzy Graphs

In this section, we will give the definition of (2,1)-fuzzy graph. The purpose of presenting this concept is to expand the rating space of intuitive fuzzy graphs and to create a suitable environment for modeling and analyzing some real-life problems.

In addition, concepts such as subgraph, isolated vertex, path and bridge of this graph are also be defined.

Definition 10. *Let B be a (2,1)-fuzzy set on a non-empty set W. A (2,1)-fuzzy set relation C on B is a mapping $C : B \to B$ such that*

$$\mu_C(v,z) \leq \mu_B(v) \wedge \mu_B(z)$$
$$\nu_C(v,z) \leq \nu_B(v) \vee \nu_B(z)$$

for every $v, z \in W$. C is a (2,1)-fuzzy relation that is defined by mapping in W as well $C : W \times W \to [0, 1]$.

Definition 11. *A (2,1)-fuzzy graph $G_{(2,1)F}$ on a non-empty set W is a pair $G_{(2,1)F} = (B, C)$, where $B : W \to [0, 1]$ is a (2,1)-fuzzy set on the set W. $C : W \times W \to [0, 1]$ is a (2,1)-fuzzy relation in W such that*

$$\mu_C(v, z) \leq \mu_B(v) \wedge \mu_B(z)$$
$$\nu_C(v, z) \leq \nu_B(v) \vee \nu_B(z)$$

and $0 \leq (\mu_C(v, z))^2 + \nu_C(v, z) \leq 1$ for every $(v, z) \in Z$, where $Z \subseteq W \times W$ is the set of edges.

Remark 1. *The following statements are true if $G_{(2,1)F}$ is a (2,1)-fuzzy graph on a non-empty set W:*

1. $C(v, z) = (0, 0)$ for every $v, z \in W \times W - Z$, where $Z \subseteq W \times W$ is the set of edges. B is defined as a (2,1)-fuzzy vertex set of $G_{(2,1)F}$ and C is defined as a (2,1)-fuzzy edge set.

2. C is defined as a (2,1)-fuzzy relation on B. A (2,1)-fuzzy relation C on B is symmetric if $C(v, z) = C(z, v)$ for every $v, z \in W$.

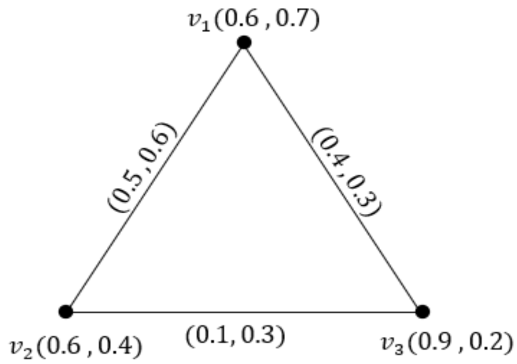

Figure 1: (2,1)-fuzzy graph

Example 1. *In the following Figure 1 represent a (2,1)-fuzzy graph.*

Definition 12. *Let $G_{(2,1)F} = (B, C)$ be a (2,1)-fuzzy graph. A (2,1)-fuzzy subgraph is a pair of $G^A_{(2,1)F} = (B^A, C^A)$ of a (2,1)-fuzzy graph $G_{(2,1)F} = (B, C)$, if $B^A \subseteq B$ and $C^A \subseteq C$ which is $\mu^A_B \leq \mu_B$, $\nu^A_B \geq \nu_B$ and $\mu^A_C(v, z) \geq \mu_C(v, z)$, $\nu^A_C(v, z) \leq \nu_C(v, z)$.*

Definition 13. *A vertex $v \in B$ in a (2,1)-fuzzy graph $G_{(2,1)F} = (B, C)$ is said to be an isolated vertex if $\mu_C(v, z) = 0$ and $\nu_C(v, z) = 0$ for all $v \in B$, $v \neq z$. That is $N(v) = \varnothing$.*

Definition 14. *If either of the two conditions is true, a path in a (2,1)-fuzzy graph is a sequence of distinct vertices $v_1, v_2, ..., v_n$:*
1. $\mu_C(v_i, v_j) > 0$ and $\nu_C(v_i, v_j) \leq 0$ for some i and j,
2. $\mu_C(v_i, v_j) \leq 0$ and $\nu_C(v_i, v_j) < 0$ for some i and j.

Definition 15. *The length of a path $P = v_1 v_2 ... v_{n+1} (n > 0)$ in $G_{(2,1)F}$ is n.*

Definition 16. *An edge (v_i, v_j) in a (2,1)-fuzzy graph if removing on edge in $G^*_{(2,1)F} = (W, Z)$ weakens the connection between a certain pair of vertices in $G^*_{(2,1)F}$ then edge considered a bridge.*

4 Domination in (2,1)-Fuzzy Graphs

In this section, the concepts of cardinality, degree, neighborhood, independent and domination of (2,1)-fuzzy graphs are defined and some examples are given regarded with this definitions. Firstly, let us define edge and vertex cardinality. These concepts will also used in the application part.

Definition 17. *Let $G_{(2,1)F} = (B, C)$ be a (2,1)-fuzzy graph on W then the vertex cardinality of $G_{(2,1)F}$ is indicated by the symbol $|B|$ and is defined as follows:*

$$|B| = \sum_{v_i \in W} \frac{1+\mu_B(v_i)-\nu_B(v_i)}{2}$$

Definition 18. *Let $G_{(2,1)F} = (B, C)$ be a (2,1)-fuzzy graph on W then the edge cardinality of $G_{(2,1)F}$ is indicated by the symbol $|C|$ and is defined as follows:*

$$|C| = \sum_{v_i, v_j \in W} \frac{1+\mu_C(v_i,v_j)-\nu_C(v_i,v_j)}{2}$$

Definition 19. *Let $G_{(2,1)F}$ be a (2,1)-fuzzy graph. Then the cardinality of $G_{(2,1)F}$ is indicated by the symbol $G_{G_{(2,1)F}}$ and is defined as follows:*

$$G_{G_{(2,1)F}} = \sum_{v_i \in W} \frac{1+\mu_B(v_i)-\nu_B(v_i)}{2} + \sum_{v_i, v_j \in W} \frac{1+\mu_C(v_i,v_j)-\nu_C(v_i,v_j)}{2}$$

Example 2. *Consider a (2,1)-fuzzy graph $G_{(2,1)F}$ on $W = \{v_1, v_2, v_3, v_4\}$ as shown in Figure 2.*

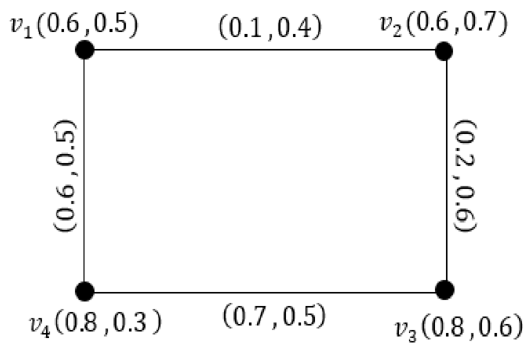

Figure 2: (2,1)-fuzzy graph

The vertex cardinality of $G_{(2,1)F}$ is 2.35 and the edge cardinality is 1.8. The cardinality of $G_{(2,1)F}$ is 4.15.

Definition 20. *The sum of the weights of strong edges incident to a vertex is degree in a (2,1)-fuzzy graph $G_{(2,1)F}$ known as the degree of the vertex and showed as $d_{G_{(2,1)F}}(v)$. The minimum degree of $G_{(2,1)F}$ is $\delta(G_{(2,1)F}) = min\{d_{G_{(2,1)F}}(v)|v \in W\}$. The maximum degree of $G_{(2,1)F}$ is $\Delta(G_{(2,1)F}) = max\{d_{G_{(2,1)F}}(v)|v \in W\}$.*

Definition 21. *If either of the two conditions is true, two vertices v_i and v_j in a (2,1)-fuzzy graph $G_{(2,1)F}$ are defined as neighbors.*
1. $\mu_C(v_i, v_j) > 0$ *and* $\nu_C(v_i, v_j) \leq 0$
2. $\mu_C(v_i, v_j) \geqslant 0$ *and* $\nu_C(v_i, v_j) < 0$, $v_i, v_j \in W$.

Definition 22. *A (2,1)-fuzzy graph with edge (v,z) is said to have a strong edge if both $\mu_C(v, z) \geqslant (\mu_C)^\infty(v, z)$ and $\nu_C(v, z) \geqslant (\nu_C)^\infty(v, z)$.*

Definition 23. *If $\mu_C(v,z) = 0$ and $\nu_C(v,z) = 0$, for all $z \in W$, $v \neq z$, then a vertex v in a (2,1)-fuzzy graph $G_{(2,1)F}$ is said to be an isolated vertex. Thus, no vertex in $G_{(2,1)F}$ can be dominated by a vertex that is isolated.*

Definition 24. *Let $G_{(2,1)F}$ represent a (2,1)-fuzzy graph on a set W that is not empty. If there is a strong edge between v and z, then v dominates z in $G_{(2,1)F}$.*

Remark 2. *The following statements are true if $G_{(2,1)F}$ is a (2,1)-fuzzy graph on a non-empty set W*

1. Domination is a symmetric relation on W for any pair of $v, z \in W$ where if v dominates z, z also dominates v.

2. $N(v)$ is precisely the set of all vertices in W that are dominated by v for any $v \in W$.

3. The dominating set of $G_{(2,1)F}$ is W itself if $\mu_C(v,z) < (\mu_C)^\infty(v,z)$ and $\nu_C(v,z) < (\nu_C)^\infty(v,z)$ for all $v, z \in W$.

4. No other vertex of $G_{(2,1)F}$ is dominated by an isolated vertex.

Definition 25. *If there is a subset S such that v dominates z for every $z \in W - S$ then that subset is said to be a dominating set in $G_{(2,1)F}$.*

Definition 26. *A dominating set S of a (2,1)-fuzzy graph, if no proper subset of S is a dominating set then $G_{(2,1)F}$ is said to be a minimal dominating set.*

Definition 27. *Lower domination number of $G_{(2,1)F}$ is the minimum cardinality among all minimal dominating sets and is represented by $\gamma_d(G_{(2,1)F})$.*

Definition 28. *Upper domination number of $G_{(2,1)F}$ is the maximum cardinality among all minimal dominating sets and is represented by $\gamma_D(G_{(2,1)F})$.*

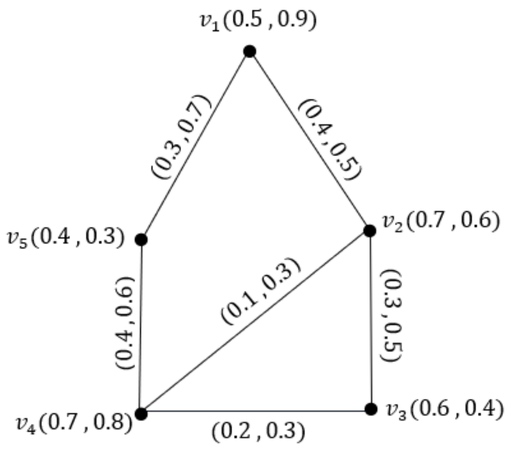

Figure 3: Domination of (2,1)-fuzzy graph

Example 3. *Consider a (2,1)-fuzzy graph $G_{(2,1)F}$ on $W = \{v_1, v_2, v_3, v_4, v_5\}$ as shown in Figure 3. In Figure 3, $\{v_2, v_5\}, \{v_1, v_3, v_4\}, \{v_1, v_2, v_5\}, \{v_1, v_2, v_3, v_4\}, \{v_1, v_3, v_4, v_5\}$ are dominating sets but $\{v_1, v_4, v_5\}$ is not a dominating set. Furthermore, among all minimal dominating sets, $\{v_2, v_5\}$ has the minimum cardinality, so $\gamma_d(G_{(2,1)F}) = 2$. Other minimal dominating sets are $\{v_1, v_3, v_4\}, \{v_1, v_2, v_5\}, \{v_1, v_2, v_4\}$. Given that among all minimal dominating sets, $\{v_1, v_2, v_4\}$ has the maximum cardinality, $\gamma_D(G_{(2,1)F}) = 3$.*

Definition 29. *Let $G_{(2,1)F}$ be a (2,1)-fuzzy graph on W without isolated vertices. If there is a vertex v, $v \in S$, $v \neq z$ such that v dominates z for every vertex $z \in W$ then the subset $S \subseteq W$ is a total dominating set.*

Definition 30. *If no proper subset of S is a total dominating set then S of a (2,1)-fuzzy graph* $G_{(2,1)F}$ *is said to be a minimal total dominating set.*

Definition 31. *The lower total domination number* $G_{(2,1)F}$ *is the minimum cardinality of a minimal total dominating set and it is represented by* $\gamma_t(G_{(2,1)F})$.

Definition 32. *The upper total domination number* $G_{(2,1)F}$ *is the maximum cardinality of a minimal total dominating set and it is represented by* $\gamma_T(G_{(2,1)F})$.

Example 4. *In Figure 3,* $\{v_2, v_3, v_4, v_5\}$ *and* $\{v_1, v_2, v_5\}$ *are minimal total dominating sets. Furthermore, among all minimal total dominating sets,* $\{v_1, v_2, v_5\}$ *has the minimum cardinality, so* $\gamma_t(G_{(2,1)F}) = 3$. *Given that among all minimal total dominating sets,* $\{v_2, v_3, v_4, v_5\}$ *has the maximum cardinality,* $\gamma_T(G_{(2,1)F}) = 4$.

Definition 33. *If there is not a strong edge connecting two vertices in a (2,1) fuzzy graph* $G_{(2,1)F}$, *they are said to be independent.*

Definition 34. *Let* $G_{(2,1)F}$ *be a (2,1)-fuzzy graph on W. A subset S of W is said to be an independet set if* $\mu_C(v,z) < (\mu_C)^\infty(v,z)$ *and* $\nu_C(v,z) < (\nu_C)^\infty(v,z)$, *for all* $v, z \in S$.

Definition 35. *If for every vertex* $v \in W - S$, *the set* $S \cup \{v\}$ *is not an independent set then the independent set S of the (2,1)-fuzzy graph* $G_{(2,1)F}$ *is said to be maximal independent set.*

Definition 36. *Lower indepence number of* $G_{(2,1)F}$ *is the minimum cardinality among all maximal independent sets and it is represented by* $\gamma_i(G_{(2,1)F})$.

Definition 37. *Upper indepence number of* $G_{(2,1)F}$ *is the maximum cardinality among all maximal independent sets and it is represented by* $\gamma_I(G_{(2,1)F})$.

Theorem 1. *A minimal dominating set is a dominating set S of a (2,1)-fuzzy graph* $G_{(2,1)F}$ *if and only if one of the following requirements is true for each* $s \in S$:

1.No vertex in S has s as a strong neighbor.

2.$N(v) \cap S = \{s\}$ *for every vertex* $v \in W - S$.

Proof. If S is a minimal dominating set of $G_{(2,1)F}$, then $S - \{s\}$ is not a dominating set for any vertex $s \in S$. Accordingly, $v \in W - (S - \{s\})$ exists and is not dominated by any vertex in $S - \{s\}$. A vertex in S does not have v as a strong neighbor if $v = s$. If v $\neq$ s, then $S - \{v\}$ dominates v instead of v being dominated by S. In this case, $N(v) \cap S = \{s\}$, meaning that the vertex v is only a strong neighbor to s in S. On the other hand, suppose that S is a dominating set such that one of the two requirements holds for each vertex $s \in S$. In the event that S is not a minimal dominating set, a dominating set $S - \{s\}$ will exist if there is a vertex $s \in S$. Given that s is a strong neighbor of at least one vertex in $S - \{s\}$, condition 1 is not satisfied. Every vertex in $W - S$ is a strong neighbor to at least one vertex in $S - \{s\}$ if $S - \{s\}$ is a dominating set. This means that condition 2 does not apply, which goes against our presumption that at least one of the criteria is true. Therefore S is a minimal dominating set.

□

Theorem 2. *If* $G_{(2,1)F}$ *is a (2,1)-fuzzy graph with no isolated vertices and S is its smallest dominating set, then* $W - S$ *is a dominating set of* $G_{(2,1)F}$.

Proof. Let S be a minimal dominating set and $s \in S$. Given that $G_{(2,1)F}$ has no isolated vertices, a vertex $v \in N(s)$ exists and at least one vertex in $S - \{s\}$ must dominated v. Hence $S - \{s\}$ is a dominating set According to Theorem 1, $v \in W - S$. Thus, $W - S$ is a dominating set since every vertex in S is dominated by at least one vertex in $W - S$. □

Definition 38. *Let $G_{(2,1)F}$ be a (2,1)-fuzzy graph on W. If $\mu_C(v,z) = \mu_B(v) \wedge \mu_B(z)$ or $\nu_C(v,z) = \nu_B(v) \vee \nu_B(z)$, then the edge (v,z) is said to be effective. Next, a (2,1)-fuzzy graph if every edge in Z is effective, then $G_{(2,1)F}$ is strong; if we have $\mu_C(v,z) = \mu_B(v) \wedge \mu_B(z)$ or $\nu_C(v,z) = \nu_B(v) \vee \nu_B(z)$ for every $v, z \in W$, then $G_{(2,1)F}$ is complete. If there exists a z such that (v,z) is an effective edge for each v, then S is a dominating set of (2,1)-fuzzy graph $G_{(2,1)F}$.*

$\alpha_E(G_{(2,1)F}) = min\{\sum_{v\in W} \frac{1+\mu_B(v)-\nu_B(v)}{2} |$ *S is effective edge dominating set of (2,1)-fuzzy graph* $\}$.

Definition 39. *The vertex set of a (2,1)-fuzzy graph $G_{(2,1)F}$ can be divided into two non-empty sets, W_1 and W_2, such that $\mu_C(v,z) = \nu_C(v,z) = 0$ if $(v,z) \in W_1$ or $(v,z) \in W_2$, then the graph is said to be bipartite. $G_{(2,1)F}$ is defined as a complete fuzzy bipartite graph if $\mu_C(v,z) = \mu_B(v) \wedge \mu_B(z)$ and $\nu_C(v,z) = \nu_B(v) \vee \nu_B(z)$ for any $v \in W_1$ and $z \in W_2$. In (2,1)-fuzzy graph $G_{(2,1)F}$, the validity of edge (v,z) is indicated by*

$$I(v,z) = (I_\mu(v,z), I_\nu(v,z)) = \left(\frac{\mu_C(v,z)}{\mu_B(v)\wedge\mu_B(z)}, \frac{\nu_C(v,z)}{1+\nu_B(v)\vee\nu_B(z)}\right).$$

If $I_\mu(v,z) \geq \frac{1}{2}$ and $I_\nu(v,z) \leq \frac{1}{2}$, the edge (v,z) is valid; if not, it is called invalid. If there is a $z \in S$ such that (v,z) is a valid edge for each $v \in W - S$, then $S \subseteq W$ is a dominating set of (2,1)-fuzzy graph $G_{(2,1)F}$.

$\alpha_v(G_{(2,1)F}) = min\{\sum_{v\in W} \frac{1+\mu_B(v)-\nu_B(v)}{2} |$ *S is valid edge dominating set of (2,1)-fuzzy graph* $\}$.

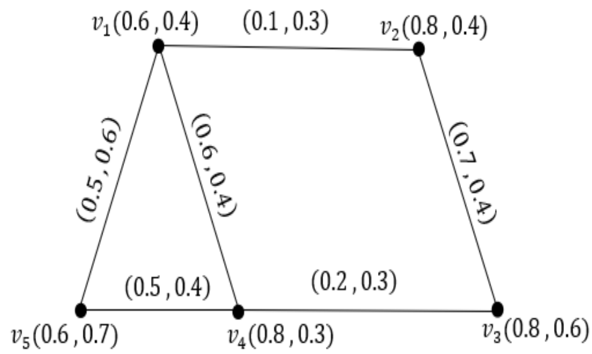

Figure 4: *Valid edges of (2,1)-fuzzy graph*

Example 5. *Consider a (2,1)-fuzzy graph $G_{(2,1)F}$ on $W = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $Z = \{(v_1,v_2), (v_1,v_3), (v_1,v_4),$
$(v_3,v_4), (v_4,v_5), (v_2,v_5)\}$.*

A quick computation confirms that $\{(v_1,v_4), (v_1,v_5), (v_4,v_5), (v_2,v_3)\}$ are valid edges but $\{(v_1,v_2), (v_3,v_4)\}$ are invalid. The membership function values are shown in Figure 4.

Dominating sets are $S_1 = \{v_1,v_2\}$, $S_2 = \{v_2,v_4\}$, $S_3 = \{v_2,v_5\}$, $S_4 = \{v_1,v_3\}$, $S_5 = \{v_3,v_5\}$, $S_6 = \{v_3,v_4\}$. Calculations are made taking into account the values of membership functions.

As a result, we confirm that $\alpha_v(G_{(2,1)F}) = 1.05$ and that the minimum dominating set is $S_5 = \{v_3,v_5\}$.

5 Application

It is planned to select a group of managers by the District Governorship in order to identify the problems of the citizens living in the districts, to provide uninterrupted services quickly and with high quality, and to solve the current problems and demands as soon as possible. Every manager must have certain characteristics. It is aimed that the managers in the selected group will do this job in the best and fastest way. An attempt is made to form a group so that each manager who is not in the group has something in common with the other manager in the group. A group with as few managers as possible should be formed for this responsibility. Since we are looking for the set with the minimum number of elements here, we can get help from the definition of dominance.

There are some good and some bad leadership qualities that distinguish managers from each other. Each vertex will be assigned values calculated based on the managers characteristics. Some criteria need to be taken into account when giving points to managers.

1. Recognition, reliability
2. Honesty, justice
3. Openness to innovation, age
4. Communication

These criteria are determined as above. Here, the coefficients of the criteria are considered as 2, 2, 1, 1, respectively, according to their degree of importance. Bad leadership traits are considered to be less recognition, less honesty, lack of innovation, and poor communication skills. These qualities should be fuzzy. Here we use the (2,1)-fuzzy graph definition to define these concepts more clearly. The items were evaluated according to the (2,1)-fuzzy graph definition and the values of the vertices in the [0,1] range were calculated. Here we use the following formula when calculating vertex values:

$$\mu(v) = \tfrac{2\mu_1(v)+2\mu_2(v)+\mu_3(v)+\mu_4(v)}{6},$$

$$\nu(v) = \tfrac{2\nu_1(v)+2\nu_2(v)+\nu_3(v)+\nu_4(v)}{6}.$$

The values of the edges are calculated as the common leadership trait and non-shared leadership trait of the two managers. In order for an edge to exist between two vertices, two people must have a connection between them. When calculating the edges, we calculate them in accordance with the definition of the (2,1)-fuzzy graph and check whether there are common features. Finally, we define vertex and edge values.

Consider a (2,1)-fuzzy graph $G_{(2,1)F}$ in the Figure 5. We have 10 managers here.

$$W = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$$

refers to the vertices, i.e., the managers, while

$$Z = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}, z_{11}, z_{12}, z_{13}, z_{14}, z_{15}\}$$

refers to the edges, that is their relationships. Here we express the relationship between v_1 and v_2 with edge z_1. In the edge values, for example, there are 20 common leadership traits and 30 non-common leadership traits between v_1 and v_2 managers. From the valid edge definition, (v_1, v_2), (v_1, v_5), (v_3, v_8) and (v_6, v_8) are invalid edges in this graph .

It is possible to talk about more than one dominating set. We need to find the minimum dominating set that is the best result for us. Some of the minimal dominating sets are as follows:

$S_1 = \{v_3, v_6, v_{10}\}$, $S_2 = \{v_4, v_6, v_7, v_8\}$, $S_3 = \{v_1, v_3, v_9, v_{10}\}$, $S_4 = \{v_1, v_2, v_4, v_{10}\}$ and $S_5 = \{v_4, v_6, v_7, v_8\}$.

As a result, we confirm that $\alpha_v(G_{(2,1)F}) = 2$ and that the minimum dominating set is $S_1 = \{v_3, v_6, v_{10}\}$.

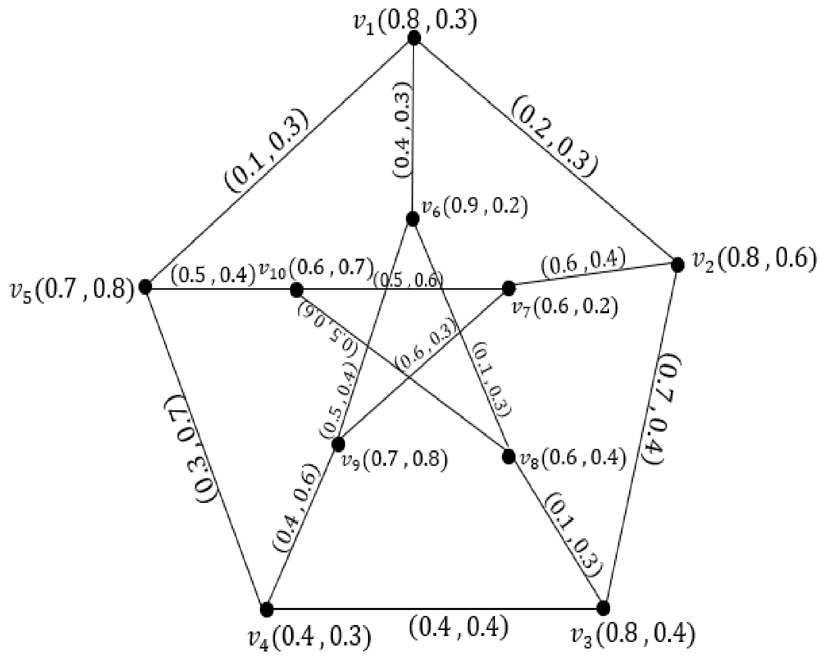

Figure 5: *(2,1)-fuzzy graph of manager values*

The algorithm of the procedure used in the application is given below.

Algorithm

Step-1. Write all the vertex values in the setting of (2,1)-fuzzy B set of managers $v_1, v_2, ..., v_n$.

Step-2. Write all the edges following the (2,1)-fuzzy graph such that $\mu_C(v_i, v_j) \leq \mu_B(v_i) \wedge \mu_B(v_j)$ and $\nu_C(v_i, v_j) \leq \nu_B(v_i) \vee \nu_B(v_j)$, $i \neq j$.

Step-3. Compare the values of the edges using the valid edge definition and find invalid edges.

Step-4. Create $S_i \subseteq W$ dominating set of vertices.

Step-5. If $\cup_j\{v_j\} = W - S_i$, then S_i is a dominating set if not S_i is not a dominating set.

Step-6. Find all of the dominating sets S of W by repeating steps 2 through 5.

Step-7. If $S_i \subseteq W$ is a minimal dominating set, it is taken as the optimal result.

Step-8. Result.

6 Conclusion

In this study, we define the (2,1)-fuzzy graph concept and aim to broaden the understanding of graph theory principles in a (2,1)-fuzzy environment. We defined the concept of domination in (2,1)-fuzzy graphs. We have created some definitions of this concept of domination and examples of these definitions. Thus, we expanded the concept of domination in intuitionistic fuzzy graphs. Finally, a numerical example and the algorithm of this example are given to explain the calculation of domination in a (2,1)-fuzzy graph in a specific real-life application. In future studies, some properties and domination types of the (2,1)-fuzzy graph can also be examined.

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